Math 275D Lecture 24 Notes

Daniel Raban

December 2, 2019

1 Integrating With Respect to Random Processes

1.1 Integrating with respect to random processes

The 1-dimensional version of Itô's formula says that

$$df(t, B_t) = f_t\, dt + f_x\, dB_t + \frac{1}{2} f_{xx}\, dt.$$

For a Brownian motion $B_t = (B_1, \dots, B_d)$ in $\mathbb{R}^d$, we have

$$df(t, B_t) = f_t\, dt + \nabla f \cdot dB + \frac{1}{2} \Delta f\, dt.$$

Let's cover something more general.

Suppose we have a process

$$X_t = X_0 + \int_0^t a(\omega, s)\, ds + \int_0^t b(\omega, s)\, dB_s.$$

Here, $a(t), b(t) \in \mathcal{F}_t$ for any t, where $\mathcal{F}_t$ is the filtration with respect to Brownian motion. We can write this as $dX_t = a(t)\, dt + b(t)\, dB_t$. How can we define $\int_0^t f(s)\, dX_s$? And in $\mathbb{R}^d$, what if we have $dX_t = a(t)\, dt + \widehat{b}(t)\, dB(t)$, where X_t, a are vectors and $\widehat{b}$ is a matrix?

To figure out the value of $\int_0^t f(s)\, dX_s$, we can write

$$df(t, X_t) = f_t\, dt + f_x\, dX_t + \frac{1}{2} f_{xx} b^2(t)\, dt,$$

where

$$f_x\, dX_t = (f_x a(t))\, dt + b(t)\, dB_t.$$

If we replace X_t with B_t, we get $a(t) = 0$ and $b(t) = 1$; but this does not really help us understand this generalization.

Here is why the formula looks like this: we have $f_t \Delta t + f_x \Delta X_t + \frac{1}{2} f_{xx} (\Delta t)^2$. How do we understand the last term? Look at $\mathbb{E}\left[\frac{(\Delta X_t)^2}{\Delta t}\right]$. As $\Delta t \to 0$, we have $\Delta X_t \to a(t)\Delta t + b(t)\Delta B_t$. So

$$\frac{(\Delta X_t)^2}{\Delta t} \to b^2(t) \cdot \frac{(\Delta B_t)^2}{\Delta t}.$$

So the last term in our formula is actually like $\frac{1}{2} f_{xx} (dX_t)^2$.

Example 1.1. Let $X_t = \int \cos(t)\, dB_t$. What is $\sin(X_t)$? This is a question from an interview book.

Returning to the vector version $dX_t = a(t)\, dt + \widehat{b}(t) dB_t$, we have

$$df(t, X_t) = f_t\, dt + \nabla \cdot dx_t + \frac{1}{2}\, dX_t \cdot \mathrm{Hess}(f)\, dX_t.$$

1.2 Quadratic variation

If we know X_t, can we find $\widehat{b}(t)$? The idea is that X_t is what you actually observe, and we are modeling it as Brownian motion. Then we want to recover some information about the model. If X_t is our process, then we want to find the **quadratic variation** $\langle X \rangle_t$:

$$\lim_{\Delta_t \downarrow 0} \sum_k (X_{t_k} - X_{t_{k-1}})^2.$$

We find that

$$\langle X \rangle_t = \int_0^t b^2(s)\, ds.$$

Then

$$\frac{d\langle X \rangle_t}{dt} = b^2(t).$$

In applications, we can assume that b is a.s. non-random. This gives us a lot of information about how the sample paths of our process vary. This is used in financial mathematics to "observe" the variation of stock prices.

1.3 Dyson spheres

We will discuss Dyson Brownian motion next time. Dyson is famous in science fiction circles for a different idea. He thought that in any developed culture, there will be a race towards more efficient forms of energy production. Solar energy if one of the most efficient and long-lasting forms of energy, so he thought that people would build solar panels close to their sun, where they can get the most energy. So eventually, the sun would be covered by solar panels, in what is called a **Dyson sphere**. Dyson proposed that to look for alien life, we should look for stars where the brightness has been reduced (to indicate that it has been covered by solar panels.